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A modified Toda spectral problem and its hierarchy of bi-Hamiltonian lattice equations

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Abstract
Starting from a modified Toda spectral problem, a hierarchy of generalized Toda lattice equations with two arbitrary constants is constructed through discrete zero curvature equations. It is shown that the hierarchy possesses a bi-Hamiltonian structure and a hereditary recursion operator, which implies that there exist infinitely many common commuting symmetries and infinitely many common commuting conserved functionals. Two cases of the involved constants present two specific integrable sub-hierarchies, one of which is exactly the Toda lattice hierarchy.

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1. Introduction

It is an important subject to find new nonlinear integrable equations, both continuous and discrete. The discovery of the inverse scattering transform (IST) brings a powerful idea to analyse nonlinear integrable equations analytically, and the IST can also lead to the geometrical and algebraic interpretation of many integrable properties of nonlinear equations such as Hamiltonian structures and Virasoro algebras [1]. The key of the IST theory is spectral problems or the Lax pairs, which pay a crucial role in the treatment of nonlinear integrable equations [2]. Other beautiful theories on integrable equations such as Sato τ-function theory [3] and R-matrix theory [4] also underscore the importance of spectral problems or the Lax pairs.

Among the well-known integrable equations are the KP-type continuous and discrete equations. The related KP theories are systematic and the residue technique can be used to present sufficiently many conserved densities required in exploring integrability of the KP equations. However, for non-KP-type equations, the working techniques are varied [5]. The structures that integrable equations possess are very particular and the involved nonlinearity

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causes much difficulty in the process of construction. It is very challenging to get new integrable equations by any method.

We will focus on lattice equations within the framework of the Lax pairs. A lattice equation

$$u_t = K(u, Eu, E^{-1}u, \ldots)$$

is said to possess a Lax pair, U and V, if it can be written as a compatibility condition (called a discrete zero curvature equation [6])

$$U_t = (EV)U - UV$$

of a discrete spatial spectral problem

$$E\phi = U(u, \lambda)\phi$$

and an associated temporal spectral problem

$$\phi_t = V(u, Eu, E^{-1}u, \ldots; \lambda)\phi$$

where λ is a spectral parameter. In the setting up of the Lax theory, the starting point is a spatial spectral problem. The equations resulting from spectral problems often possess bi-Hamiltonian structures and hereditary recursion operators [7]. Therefore, infinitely many symmetries and infinitely many conserved densities can be guaranteed, which exhibit integrability of the equations under consideration.

In this paper, starting from a modified Toda spectral problem, we would like to construct a hierarchy of generalized Toda lattice equations with two arbitrary constants through the discrete Lax technique or the technique of discrete zero curvature equations. The bi-Hamiltonian theory on integrable equations [8, 9] will be used to establish a bi-Hamiltonian formulation and to present a hereditary recursion operator for the resulting lattice hierarchy. Thus, there exist infinitely many common symmetries and infinitely many common conservation laws. It will also be shown that two cases of the involved constants present two specific sub-hierarchies of integrable lattice equations, one of which is exactly the Toda lattice hierarchy [8].

2. A discrete spectral problem and related lattice equations

Let f be a lattice function. We specify the shift operator E and the inverse E^{-1} of E as

$$(Ef)(n) = f(n+1) \qquad (E^{-1}f)(n) = f(n-1) \qquad n \in \mathbb{Z} \tag{2.1}$$

and always write

$$f^{(k)} = E^k f \qquad k \in \mathbb{Z}. \tag{2.2}$$

Of course, the shift unit could also be taken as some other number, less than or greater than one. Let us now introduce the following discrete spectral problem,

$$E\phi = U(u, \lambda)\phi \qquad U(u, \lambda) = \begin{bmatrix} 0 & 1 \\ (\alpha\lambda + \beta)r & \lambda + s \end{bmatrix} \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad u = \begin{bmatrix} r \\ s \end{bmatrix} \tag{2.3}$$

where λ is a spectral parameter, and α and β are two constants satisfying the nondegenerate condition

$$\alpha^2 + \beta^2 \neq 0. \tag{2.4}$$

This discrete spectral problem is a modified Toda spectral problem. If $\alpha = 0$ and $\beta = 1$, then (2.3) is exactly the Toda spectral problem [8]. The discrete spectral problem is also equivalent to

$$(E - \beta r E^{-1} - s)\psi = \lambda(\alpha r E^{-1} - 1)\psi \qquad \psi = E\phi_1.$$

This is one of the generalized eigenvalue problems of the type

$$L_1(u)\psi = \lambda L_2(u)\psi$$

where $L_1(u)$ and $L_2(u)$ are two difference (or differential–difference) operators and u is the potential. Choose $L_2(u)$ to be the identity operator and $L_1(u)$ to be a Lax operator with infinitely many fields

$$L_1(u) = u_n E^n + u_{n-1} E^{n-1} + \cdots$$

where n is an integer and the coefficients u_i are a sequence of field variables with possibly non-commuting components, and then we have a discrete eigenvalue problem of the standard KP type. Lax operators with finitely many fields are special reductions of this Lax operator with infinitely many fields. When $\alpha \neq 0$, $L_2(u) = \alpha r E^{-1} - 1$ in our case is not a constant operator, and thus the above spectral eigenvalue problem, equivalent to (2.3), is not of the standard KP type.

In order to construct the associated lattice equations, we first solve the stationary discrete zero curvature equation

$$(E\Gamma)U - U\Gamma = 0. \tag{2.5}$$

If we choose

$$\Gamma = \begin{bmatrix} a & b \\ (\alpha\lambda + \beta)c & -a \end{bmatrix} \tag{2.6}$$

then equation (2.5) becomes

$$\begin{cases} rb^{(1)} - c = 0 \\ (a^{(1)} + a) + \lambda b^{(1)} + s b^{(1)} = 0 \\ \alpha\lambda(c^{(1)} - rb) + \beta(c^{(1)} - rb) - \lambda(a^{(1)} - a) - s(a^{(1)} - a) = 0. \end{cases} \tag{2.7}$$

Taking the selection

$$a = \sum_{m=0}^{\infty} a_m \lambda^{-m} \qquad b = \sum_{m=0}^{\infty} b_m \lambda^{-m} \qquad c = \sum_{m=0}^{\infty} c_m \lambda^{-m} \tag{2.8}$$

for (2.7) and comparing each power of λ in the equations of (2.7), we obtain the initial set of relations:

$$b^{(1)} = 0 \qquad c_0 = 0 \qquad a_0 - a_0^{(1)} = -\alpha\left(c_0^{(1)} - rb_0\right) = \alpha r b_0 \tag{2.9}$$

and the recursion relation:

$$\begin{cases} c_{m+1} - r b_{m+1}^{(1)} = 0 \qquad m \geqslant 0 \\ b_{m+1}^{(1)} + s b_m^{(1)} + \left(a_m^{(1)} + a_m\right) = 0 \qquad m \geqslant 0 \\ \left(a_{m+1}^{(1)} - a_{m+1}\right) + s\left(a_m^{(1)} - a_m\right) + \alpha\left(r b_{m+1} - c_{m+1}^{(1)}\right) + \beta\left(r b_m - c_m^{(1)}\right) = 0 \qquad m \geqslant 0. \end{cases} \tag{2.10}$$

Upon fixing initial data in (2.9):

$$a_0 = -\frac{1}{2} \qquad b_0 = 0 \tag{2.11}$$

the recursion relation (2.10) uniquely determines the lattice functions a_m, b_m and c_m, $m \geqslant 1$, under the initial conditions

$$a_1|_{u=0} = c_1|_{u=0} = 0 \qquad a_m|_{u=0} = b_m|_{u=0} = c_m|_{u=0} = 0 \qquad m \geqslant 2. \tag{2.12}$$

Different initial data and initial conditions will only lead to linear combinations of the resulting lattice functions. The first few lattice functions are easily computed:

$$\begin{cases} a_1 = \alpha r \qquad b_1 = 1 \qquad c_1 = r \\ a_2 = -\alpha^2 r^{(1)} r - \alpha^2 r^2 - \alpha^2 r r^{(-1)} - \alpha r s - \alpha r s^{(-1)} + \beta r \\ b_2 = -\alpha r - \alpha r^{(-1)} - s^{(-1)} \qquad c_2 = -rs - \alpha r^2 - \alpha r r^{(1)}. \end{cases} \tag{2.13}$$

Moreover, from (2.5), we can know [10] that $(E-1)\mathrm{tr}(\Gamma^k) = 0$ for all $k \geqslant 1$. In particular, $\mathrm{tr}(\Gamma^2) = 2[a^2 + (\alpha\lambda+\beta)bc]$ is a constant, and let us say γ. Then, we have a recursion relation for a_m:

$$a_{m+1} = \sum_{i=1}^{m} a_i a_{m-i+1} + \alpha \sum_{i=1}^{m+1} b_i c_{m-i+2} + \beta \sum_{i=1}^{m} b_i c_{m-i+1} - \frac{1}{2}\gamma \qquad m \geqslant 1. \tag{2.14}$$

This, together with the first two recursion relations in (2.10), implies through the mathematical induction that all lattice functions a_m, b_m and c_m, $m \geqslant 1$, are local and they are just difference polynomials in r and s.

As usual, we choose that

$$\begin{aligned} V_m &= \begin{bmatrix} (\lambda^m a)_+ & (\lambda^m b)_+ \\ (\alpha\lambda+\beta)(\lambda^m c)_+ & -(\lambda^m a)_+ \end{bmatrix} \\ &= \begin{bmatrix} \displaystyle\sum_{i=0}^{m} a_i \lambda^{m-i} & \displaystyle\sum_{i=0}^{m} b_i \lambda^{m-i} \\ (\alpha\lambda+\beta)\displaystyle\sum_{i=0}^{m} c_i \lambda^{m-i} & -\displaystyle\sum_{i=0}^{m} a_i \lambda^{m-i} \end{bmatrix} \end{aligned} \qquad m \geqslant 0. \tag{2.15}$$

Then it follows from (2.10) that

$$(EV_m)U - UV_m = \begin{bmatrix} 0 & -b_{m+1}^{(1)} \\ (\alpha\lambda+\beta)c_{m+1} & \beta\big(c_m^{(1)} - r b_m\big) - s\big(a_m^{(1)} - a_m\big) \end{bmatrix}.$$

To generate the associated lattice equations through the discrete zero curvature equation, we need to take a modification

$$\Delta_m = \begin{bmatrix} b_{m+1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.16}$$

and define the auxiliary Lax operators

$$V^{[m]} = V_m + \Delta_m \qquad m \geqslant 0. \tag{2.17}$$

Then, a direct calculation leads to the following matrix:

$$(EV^{[m]})U - UV^{[m]} = \begin{bmatrix} 0 & 0 \\ (\alpha\lambda+\beta)(c_{m+1} - r b_{m+1}) & \beta\big(c_m^{(1)} - r b_m\big) - s\big(a_m^{(1)} - a_m\big) \end{bmatrix} \tag{2.18}$$

which is consistent with U_{t_m}. Therefore, the temporal evolution laws for the eigenfunction ϕ to obey can be taken to be

$$\phi_{t_m} = V^{[m]}\phi \qquad m \geqslant 0. \tag{2.19}$$

Then the compatibility conditions of the discrete spectral problem (2.3) and the associated spectral problems (2.19)

$$U_{t_m} = (EV^{[m]})U - UV^{[m]} \qquad m \geqslant 0 \tag{2.20}$$

give rise to the following hierarchy of lattice equations:

$$\begin{cases} r_{t_m} = c_{m+1} - rb_{m+1} & m \geqslant 0 \\ s_{t_m} = -\alpha\big(c^{(1)}_{m+1} - rb_{m+1}\big) + \big(a^{(1)}_{m+1} - a_{m+1}\big) \\ \quad = \beta\big(c^{(1)}_m - rb_m\big) - s\big(a^{(1)}_m - a_m\big) & m \geqslant 0. \end{cases} \tag{2.21}$$

Noting (2.13), we have the first nonlinear lattice equation

$$\begin{cases} r_{t_1} = r(s^{(-1)} - s) + \alpha r(r^{(-1)} - r^{(1)}) \\ s_{t_1} = \alpha s(r - r^{(1)}) + \beta(r^{(1)} - r). \end{cases} \tag{2.22}$$

Interestingly, when $\alpha = 0$ and $\beta = -1$, (2.22) becomes the Toda lattice equation [11]

$$r_{t_1} = r(s^{(-1)} - s) \qquad s_{t_1} = r - r^{(1)} \tag{2.23}$$

and when $\alpha = 1$ and $\beta = 0$, (2.22) gives the lattice equation

$$r_{t_1} = r(s^{(-1)} - s) + r(r^{(-1)} - r^{(1)}) \qquad s_{t_1} = s(r - r^{(1)}) \tag{2.24}$$

which was previously presented in [12]. We chose $\beta = -1$ but not $\beta = 1$ to get the Toda lattice equation (2.23), since a negative value is selected for a_0 in (2.11). Obviously, the lattice equation (2.24) is linearly independent of the Toda lattice equation (2.23). Namely, any linear transformation of dependent variables cannot transform one of them to the other.

To establish a Hamiltonian structure for a lattice hierarchy, we can usually use the trace identity [10], but in this case, we fail to apply it. The reason may be the rank problem of the discrete spectral problem (2.3). It is not easy to define a homogeneous rank for the terms involving $(\alpha\lambda + \beta)r$. If we observe the recursion relation (2.10) more carefully and note that a_2 is not homogeneous due to the last term βr, we find that when $m \geqslant 3$, all a_m, b_m and c_m are not homogeneous difference polynomials in r and s, indeed. However, inspired by

$$\operatorname{tr}\left((\Gamma U^{-1})\frac{\partial U}{\partial r}\right) = \frac{a}{r} \qquad \operatorname{tr}\left((\Gamma U^{-1})\frac{\partial U}{\partial s}\right) = \frac{c}{r}$$

which are the sources of conserved functionals in the trace identity, we can rewrite those lattice equations in (2.21) as

$$u_{t_m} = \begin{bmatrix} r \\ s \end{bmatrix}_{t_m} = J_1 \begin{bmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{bmatrix} \qquad m \geqslant 0 \tag{2.25}$$

where J_1 is a local difference operator defined by

$$J_1 = \begin{bmatrix} 0 & r - rE^{-1} \\ Er - r & \alpha(rE^{-1} - Er) \end{bmatrix}. \tag{2.26}$$

Now taking advantage of the recursion relation (2.10), we can easily obtain

$$\begin{bmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{bmatrix} = \Psi \begin{bmatrix} \dfrac{a_m}{r} \\ \dfrac{c_m}{r} \end{bmatrix} \qquad m \geqslant 0 \tag{2.27}$$

where the operator Ψ is given by

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ -(1+E)r & -s \end{bmatrix} \tag{2.28}$$

with two entries in the first row:

$$\begin{cases} \Psi_{11} = -\alpha(1+E)r - \dfrac{1}{r}(E-1)^{-1}[\alpha r(E-E^{-1})r - s(1-E)r] \\ \Psi_{12} = -\alpha s - \dfrac{1}{r}(E-1)^{-1}[\alpha r(1-E^{-1})s - \beta(Er - rE^{-1})]. \end{cases}$$

In practice, the inverse operator $(E-1)^{-1}$ will only be defined on those lattice functions in the range of $E-1$ as follows: if we have $g = (E-1)f$ and $f|_{u=0} = 0$, then we set $f = (E-1)^{-1}g$.

The recursion relation (2.27) motivates us to introduce another local difference operator J_2:

$$J_2 = J_1\Psi = \begin{bmatrix} -rEr + rE^{-1}r & -rs + rE^{-1}s \\ rs - sEr & \beta(Er - rE^{-1}) \end{bmatrix} \tag{2.29}$$

and then the lattice hierarchy (2.21) can be written in the following form:

$$u_{t_m} = \begin{bmatrix} r \\ s \end{bmatrix}_{t_m} = J_1 \begin{bmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{bmatrix} = J_2 \begin{bmatrix} \dfrac{a_m}{r} \\ \dfrac{c_m}{r} \end{bmatrix} \qquad m \geqslant 0. \tag{2.30}$$

At this moment, we can guess that the lattice hierarchy (2.21) has a bi-Hamiltonian formulation. This is true, indeed. In the next section, we will prove that J_1 and J_2 are a Hamiltonian pair and

$$G_m := \begin{bmatrix} \dfrac{a_{m+1}}{r} \\ \dfrac{c_{m+1}}{r} \end{bmatrix} \qquad m \geqslant 0 \tag{2.31}$$

are all gradient vector functions.

3. Bi-Hamiltonian structure

Let us first introduce the basic notion of the Hamiltonian theory of lattice equations, to achieve clarity in describing our design, procedure and results. The variational derivative, the Gateaux derivative and the inner product are defined by

$$\frac{\delta \tilde{H}}{\delta u} = \left(\frac{\delta \tilde{H}}{\delta r}, \frac{\delta \tilde{H}}{\delta s}\right)^T \qquad \frac{\delta \tilde{H}}{\delta r} = \sum_{n\in\mathbb{Z}} E^{-n}\frac{\partial H}{\partial r^{(n)}} \qquad \frac{\delta \tilde{H}}{\delta s} = \sum_{n\in\mathbb{Z}} E^{-n}\frac{\partial H}{\partial s^{(n)}} \tag{3.1}$$

$$P'(u)[v] = \frac{\partial}{\partial \varepsilon} P(u+\varepsilon v)\bigg|_{\varepsilon=0} \tag{3.2}$$

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \langle f(n), g(n)\rangle \tag{3.3}$$

where $u = (r,s)^T$, $\tilde{H} = \sum_{n\in\mathbb{Z}} H(u(n))$ is a functional, P can be either a vector function or an operator, v is a two-dimensional vector function, f and g are two-dimensional vector functions, and $\langle f(n), g(n)\rangle$ denotes the standard inner product of $f(n)$ and $g(n)$ in the Euclidean space $\mathbb{R}^2$. Denote by $J^\dagger$ the adjoint operator of an operator J with respect to the product (3.3), i.e., $J^\dagger$ is defined by

$$\langle f, J^\dagger g\rangle = \langle Jf, g\rangle. \tag{3.4}$$

Obviously, $E^{\dagger} = E^{-1}$ and $(J^{-1})^{\dagger} = (J^{\dagger})^{-1}$. If an operator J has the property $J^{\dagger} = -J$, then J is called skew-symmetric. For example, the difference operator $\Delta := E - E^{-1}$ is skew-symmetric.

Definition 3.1. *A linear operator J is called a Hamiltonian operator, if it is a skew-symmetric operator and satisfies the Jacobi identity*

$$\langle J'(u)[Jf]g, h\rangle + \text{cycle}(f, g, h) = 0. \tag{3.5}$$

A pair of operators J_1 and J_2 is called a Hamiltonian pair, if any linear combination $\alpha J_1 + \beta J_2$ with two constants α and β is Hamiltonian.

The commutator between vector functions and the associated Poisson bracket with a given Hamiltonian operator J are given by

$$[f, g] = f'(u)[g] - g'(u)[f] \tag{3.6}$$

and

$$\{\tilde{F}, \tilde{G}\}_J = \left\langle \frac{\delta \tilde{F}}{\delta u}, J\frac{\delta \tilde{G}}{\delta u} \right\rangle \qquad \tilde{F} = \sum_{n\in\mathbb{Z}} F(u(n)) \qquad \tilde{G} = \sum_{n\in\mathbb{Z}} G(u(n)) \tag{3.7}$$

respectively. Note that this Poisson bracket is defined especially through the use of the variational derivative. Two vector functions f and g are called commuting if $[f, g] = 0$. Similarly, two functionals $\tilde{F}$ and $\tilde{G}$ are called commuting under the Poisson bracket $\{\cdot\,,\cdot\}_J$ if $\{\tilde{F}, \tilde{G}\}_J = 0$. If an operator J is Hamiltonian, then we have [13]

$$\left[J\frac{\delta \tilde{F}}{\delta u}, J\frac{\delta \tilde{G}}{\delta u}\right] = J\{\tilde{F}, \tilde{G}\}_J. \tag{3.8}$$

Therefore, if $\tilde{F}$ and $\tilde{G}$ are commuting under the Poisson bracket $\{\cdot\,,\cdot\}_J$, then the corresponding vector functions $J\frac{\delta \tilde{F}}{\delta u}$ and $J\frac{\delta \tilde{G}}{\delta u}$ are commuting as well.

Definition 3.2. *If a lattice equation $u_t = K(u, Eu, E^{-1}u, \ldots)$ can be written as*

$$u_t = K(u) = J\frac{\delta \tilde{H}}{\delta u} \qquad \tilde{H} = \sum_{n\in\mathbb{Z}} H(u(n)) \tag{3.9}$$

where J is Hamiltonian, then it is called a Hamiltonian equation. If it can be written as

$$u_t = K(u) = J_1\frac{\delta \tilde{H}_1}{\delta u} = J_2\frac{\delta \tilde{H}_2}{\delta u} \qquad \tilde{H}_1 = \sum_{n\in\mathbb{Z}} H_1(u(n)) \qquad \tilde{H}_2 = \sum_{n\in\mathbb{Z}} H_2(u(n)) \tag{3.10}$$

where J_1 and J_2 are a Hamiltonian pair, then it is called a bi-Hamiltonian equation.

The bi-Hamiltonian formulation is a beautiful characteristic structure [8, 9], which often guarantees the existence of infinitely many symmetries and infinitely many conserved functionals, exhibiting integrability of the equations under consideration.

In what follows, we would like to show that the lattice hierarchy (2.21) is bi-Hamiltonian. To proceed, we now choose a specific 2×2 matrix local difference operator:

$$J(u) = \begin{bmatrix} -\sigma_1(rEr - rE^{-1}r) & -\sigma_1(rs - rE^{-1}s) + \sigma_2(r - rE^{-1}) \\ \sigma_1(rs - sEr) + \sigma_2(Er - r) & \sigma_3(rE^{-1} - Er) \end{bmatrix} \tag{3.11}$$

where $\sigma_i, 1 \leqslant i \leqslant 3$, are three arbitrary constants. The action of J is taken as the left multiplication, and thus it is linear. Note that J is itself nonlinear with respect to u.

Theorem 3.1. *The difference operator $J(u)$ defined by (3.11) is a Hamiltonian operator for all values of three constants σ_i, $1 \leqslant i \leqslant 3$.*

Proof. It is easy to see that J is skew-symmetric. The Jacobi identity can be checked through a straightforward but tedious computation, which is given in the appendix. It then follows that J is Hamiltonian whatever the three constants σ_1, σ_2 and σ_3 are. The proof is finished. □

The case of $\sigma_1 = 0, \sigma_2 = 1$ and $\sigma_3 = \alpha$ leads to the Hamiltonian operator J_1 defined by (2.26), and the other case of $\sigma_1 = 1, \sigma_2 = 0$ and $\sigma_3 = -\beta$ leads to the Hamiltonian operator J_2 defined by (2.29). Therefore, based on theorem 3.1, J_1 and J_2 are a Hamiltonian pair.

It is also not difficult to check that J_1 is nondegenerate. Namely, if there is a 1×2 matrix local difference operator $\bar{J}_1$ such that $\bar{J}_1 J_1 = 0$, then $\bar{J}_1 = 0$. Suppose that $\bar{J}_1 = (A, B)$ such that $\bar{J}_1 J_1 = 0$, where A and B are two local difference operators, then we have $B(Er - r) = 0$ which can lead to $B = 0$, and thus $A(r - rE^{-1}) = 0$ which will lead to $A = 0$. Therefore, $\bar{J}_1 = 0$. This implies that J_1 is nondegenerate. Actually, on some well-selected space of vector functions (e.g., the space spanned by all $J_1 G_m$, $m \geqslant 1$), the inverse operator of J_1 can be explicitly introduced as follows:

$$J_1^{-1} = \begin{bmatrix} \alpha(1 - E^{-1})^{-1}\dfrac{1}{r} + \dfrac{\alpha}{r}(E - 1)^{-1} & \dfrac{1}{r}(E - 1)^{-1} \\ (1 - E^{-1})^{-1}\dfrac{1}{r} & 0 \end{bmatrix} \tag{3.12}$$

where $(1 - E^{-1})^{-1} = E(E - 1)^{-1}$ and $(E - 1)^{-1}$ is defined as before. Then, it is obvious to see that the adjoint operator of Ψ, denoted by Φ, reads

$$\Phi := \Psi^{\dagger} = J_2 J_1^{-1} = \begin{bmatrix} \Phi_{11} & -r(1 + E^{-1}) \\ \Phi_{21} & -s \end{bmatrix} \tag{3.13}$$

with two entries in the first column:

$$\begin{cases} \Phi_{11} = -\alpha r(1 + E^{-1}) - \alpha r(E - E^{-1})r(1 - E^{-1})^{-1}\dfrac{1}{r} - r(1 - E^{-1})s(1 - E^{-1})^{-1}\dfrac{1}{r} \\ \Phi_{21} = -\alpha s + \alpha s(1 - E)r(1 - E^{-1})^{-1}\dfrac{1}{r} + \beta(Er - rE^{-1})(1 - E^{-1})^{-1}\dfrac{1}{r}. \end{cases}$$

Thus, $\Phi = \Psi^{\dagger}$ is a common hereditary recursion operator for the hierarchy (2.21) [14].

Now the remaining task to establish a bi-Hamiltonian structure for (2.21) is to prove the vector functions $G_m, m \geqslant 0$, defined by (2.31) are gradient. We use the bi-Hamiltonian theory [8, 9] to verify this property.

A direct computation shows that the first two vector functions G_0 and G_1 are gradient, i.e., we have

$$G_0 = \frac{\delta \tilde{H}_0}{\delta u} \qquad G_1 = \Psi G_0 = \frac{\delta \tilde{H}_1}{\delta u} \qquad \tilde{H}_0 = \sum_{n \in \mathbb{Z}} H_0(u(n)) \qquad \tilde{H}_1 = \sum_{n \in \mathbb{Z}} H_1(u(n)) \tag{3.14}$$

where the densities of two functionals are given by

$$H_0 = \alpha r + s \qquad H_1 = -\alpha rs - \frac{1}{2}s^2 - \alpha r s^{(-1)} - \frac{\alpha}{2}r^2 - \alpha^2 r r^{(-1)} + \beta r. \tag{3.15}$$

Recall we already know that J_1 and J_2 are a Hamiltonian pair, of which J_1 is nondegenerate; and that $J_1 G_{m+1} = J_2 G_m, m \geqslant 0$, where $G_m, m \geqslant 0$, are the local vector lattice functions defined by (2.31). Therefore, the bi-Hamiltonian theory (in particular, see lemma 7.25,

chapter 7 of [9]) allows us to conclude that all $G_m, m \geqslant 0$ are gradient, i.e., there exist functionals $\tilde{H}_m$ satisfying

$$G_m = \frac{\delta \tilde{H}_m}{\delta u} \qquad m \geqslant 0. \tag{3.16}$$

Now from the variational calculus [8], the functionals $\tilde{H}_m$ can be computed by

$$\tilde{H}_m = \sum_{m\in\mathbb{Z}} H_m(u(n)) \qquad H_m = \int_0^1 u^T G_m(\mu u)\,\mathrm{d}\mu \qquad m \geqslant 0 \tag{3.17}$$

where $G_m, m \geqslant 0$ are defined by (2.31).

If we define an initial functional

$$\tilde{H}_{-1} = \sum_{m\in\mathbb{Z}} H_{-1}(u(n)) \qquad H_{-1} = -\frac{\ln r}{2} \tag{3.18}$$

then, we have

$$G_0 = \Psi G_{-1} \qquad G_{-1} = \frac{\delta \tilde{H}_{-1}}{\delta u} = \left(-\frac{1}{2r}, 0\right)^T. \tag{3.19}$$

Thus $\tilde{H}_{-1}$ is also a conserved functional. G_{-1} is generated from $\ker J_2$. Let us sum up all the above results in the following theorem.

Theorem 3.2. *All lattice equations in the lattice hierarchy (2.21) have a bi-Hamiltonian formulation*

$$u_{t_m} = J_1 G_m = J_1 \frac{\delta \tilde{H}_m}{\delta u} = J_2 \frac{\delta \tilde{H}_{m-1}}{\delta u} \qquad m \geqslant 1 \tag{3.20}$$

where $J_1, J_2, G_m, m \geqslant 1, \tilde{H}_m, m \geqslant 0$ are defined by (2.26), (2.29), (2.31) and (3.17), respectively. Furthermore, the lattice hierarchy (2.21) has infinitely many common commuting symmetries $\{J_1 G_m\}_1^\infty$, and infinitely many common conserved functionals $\{\tilde{H}_m\}_{-1}^\infty$ being mutually commuting under either of the Poisson brackets $\{\cdot,\cdot\}_{J_1}$ and $\{\cdot,\cdot\}_{J_2}$, where $\tilde{H}_{-1}$ is defined by (3.18).

This theorem implies that the lattice hierarchy (2.21) is an integrable hierarchy in the Liouville sense. Note that $J_1 G_0 = J_1 G_{-1} = 0$ and so we do not include them in the set of symmetries. This also indicates that associated with the discrete spectral problem (2.3), there is no negative hierarchy with the same bi-Hamiltonian formulation as (3.20), starting from G_{-1}.

This is a different characteristic from the Ablowitz–Ladik discrete spectral problem associated with the discrete nonlinear Schrödinger equation [15]. Moreover, the bi-Hamiltonian structure presented in (3.20) is much simpler, since the Hamiltonian operators of the Ablowitz–Ladik positive and negative hierarchies possess higher-order nonlinear dependence on the potentials [16].

4. Concluding remarks

A hierarchy of generalized bi-Hamiltonian Toda lattice equations has been presented, starting from a modified Toda spectral problem. By the bi-Hamiltonian theory, all lattice equations in the resulting hierarchy have been shown to possess an infinite collection of common commuting symmetries and conserved functionals. It is particularly interesting that the obtained hierarchy contains two specific integrable sub-hierarchies as its examples. One is exactly the Toda

hierarchy and the other is the lattice hierarchy that we presented in [12]. The verification of the bi-Hamiltonian formulation relied on a matrix difference Hamiltonian operator involving three arbitrary constants.

The modified Toda spectral problem discussed involves a linear term $\alpha\lambda+\beta$ of the spectral parameter λ in front of one potential. The associated integrable lattice hierarchy connects the Toda hierarchy and the lattice hierarchy in [12] together. This is the same phenomenon as happened with the coupled AKNS-Kaup–Newell hierarchy [17]. It is interesting to see if the idea of adding a linear term $\alpha\lambda+\beta$ works for other spectral problems, while generating coupled integrable equations.

Besides the method of zero curvature equations used in our discussion, there are other techniques to construct integrable lattices, for example, the algebro–geometric approach, the Poisson algebra method, the classical R-matrix method, etc (see [18–21]). For Lax operators of the standard KP type, the multi-Hamiltonian structures can be generated from Poisson brackets associated with R-matrix structures [19, 21]. The resulting Hamiltonian structures possess homogeneous dependence on potentials, and separation of variables works for associated integrable lattices, yielding discrete Jacobi inversion problems [18]. However, for the reduced Lax operators of the standard KP type, one has to take the compatibility of reduced Lax pairs into consideration and make the Dirac reduction to construct multi-Hamiltonian structures [21]. The cases of Lax operators of non-standard KP type often need direct computations, starting from zero curvature equations.

We point out that there are also three ways to formally define the inverse of the difference operator $\Delta = E - E^{-1}$, and thus the inverses of other difference operators. We just list some of them for reference.

Method 1. Use both the right-side and the left-side limits in the computation:

$$\Delta^{-1} = (E-E^{-1})^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty}\right) E^{2k+1}$$

$$(E-1)^{-1} = (1+E^{-1})\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty}\right) E^{k}$$

$$(1-E^{-1})^{-1} = (E+1)\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty}\right) E^{k+1}$$

$$(1+E^{-1})^{-1} = (E-1)\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty}\right) (-1)^{k+1} E^{k+1}$$

$$(E+1)^{-1} = (1-E^{-1})\Delta^{-1} = \frac{1}{2}\left(\sum_{k=-\infty}^{-1} - \sum_{k=0}^{\infty}\right) (-1)^{k+1} E^{k}$$

Methods 2 and 3. Use the left-side limit or the right-side limit in the computation:

$$\Delta^{-1} = (E-E^{-1})^{-1} = \sum_{k=-\infty}^{-1} E^{2k+1} \quad \text{or} \quad -\sum_{k=0}^{\infty} E^{2k+1}$$

$$(E-1)^{-1} = (1+E^{-1})\Delta^{-1} = \sum_{k=-\infty}^{-1} E^{k} \quad \text{or} \quad -\sum_{k=0}^{\infty} E^{k}$$

$$(1-E^{-1})^{-1} = (E+1)\Delta^{-1} = \sum_{k=-\infty}^{-1} E^{k+1} \quad \text{or} \quad -\sum_{k=0}^{\infty} E^{k+1}$$

$$(1+E^{-1})^{-1} = (E-1)\Delta^{-1} = \sum_{k=-\infty}^{-1} (-1)^{k+1}E^{k+1} \quad \text{or} \quad -\sum_{k=0}^{\infty} (-1)^{k+1}E^{k+1}$$

$$(E+1)^{-1} = (1-E^{-1})\Delta^{-1} = \sum_{k=-\infty}^{-1} (-1)^{k+1}E^{k} \quad \text{or} \quad -\sum_{k=0}^{\infty} (-1)^{k+1}E^{k}.$$

These inverse operators are nonlocal and often appear in the construction of recursion operators and master symmetries of integrable equations [6]. The formulae for computing them are helpful in deriving expressions and establishing formulations related to lattice equations.

We remark that when $\alpha = 1$ and $\beta = 0$, the initial conserved densities defined by (3.15) are different from the previous ones H_0^{old} and H_1^{old}:

$$H_0^{\text{old}} = s + \frac{r}{2} + \frac{r^{(1)}}{2}$$

$$H_1^{\text{old}} = -\frac{1}{4}[2s^2 + 4rs + 4r^{(1)}s + 2r^{(1)}r + rr^{(-1)} + r^2 + r^{(2)}r^{(1)} + (r^{(1)})^2]$$

presented in [12]. But we have two relations between them

$$H_0 - H_0^{\text{old}} = (E-1)\left(-\frac{r}{2}\right)$$

$$H_1 - H_1^{\text{old}} = (E-1)\left(rs^{(-1)} + \frac{r^{(1)}r}{4} + \frac{r^2}{4} + \frac{3rr^{(-1)}}{4}\right).$$

Therefore, it follows from $\dfrac{\delta}{\delta u}((E-1)f(u)) = 0$ that when $\alpha = 1$ and $\beta = 0$, the lattice hierarchy (2.21) boils down to the lattice hierarchy presented previously in [12], indeed.

Many other problems deserve further investigation for the resulting hierarchy of coupled lattice equations (2.21), for example, Bäcklund transformations, τ-symmetries, constrained flows, the Hirota form and soliton solutions. The arbitrariness of three constants in the Hamiltonian operator (3.11) may also provide other Hamiltonian pairs, which can be used to present new integrable lattice hierarchies. More generally, a combination of the resulting Hamiltonian operator with a constant coefficient matrix difference operator would carry much more information on related integrable hierarchies of Toda type (see [22] for the KdV and the coupled KdV cases).

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Appendix. The proof of the Jacobi identity

The following is a detailed proof of the Jacobi identity:

$$\langle J'(u)[Jf]g, h\rangle + \text{cycle}(f, g, h) = 0$$

for the Hamiltonian operator defined by (3.11). Assume that

$$f = (f_1(n), f_2(n))^T \qquad g = (g_1(n), g_2(n))^T \qquad h = (h_1(n), h_2(n))^T$$

are three arbitrary vector functions, which are required to be rapidly vanishing at infinity. We combine the terms in $\langle J'(u)[Jf]g, h\rangle$ containing $\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2, \sigma_1\sigma_3$ and $\sigma_2\sigma_3$, respectively. Through a straightforward but laborious computation, we find that the coefficients of $\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2, \sigma_1\sigma_3$ and $\sigma_2\sigma_3$ read

$$\begin{aligned}
\sigma_1^2 : \sum_{n\in\mathbb{Z}}[&-r(Er)s(Ef_1)g_1h_2 + (Er)rsf_2(Eg_1)h_1 + r(E^{-1}r)s(E^{-1}f_1)g_1h_2\\
&- r(E^{-1}r)sf_2(E^{-1}g_1)h_1 - (Er)(E^2r)s(E^2f_2)(Eg_1)h_2\\
&- r(Er)(E^{-1}s)(E^{-1}f_2)(Eg_1)h_1 - r(Er)sf_1(Eg_1)h_2\\
&+ rs(Er)(Ef_1)g_2h_1 - r^2sf_2g_1h_2 + rs^2f_2g_2h_1 + rs(E^{-1}s)(E^{-1}f_2)g_1h_2\\
&- rs(E^{-1}s)f_2(E^{-1}g_2)h_1 + (Er)s(Es)(Ef_2)(Eg_1)h_2\\
&- rs(E^{-1}s)(E^{-1}f_2)g_2h_1 - s^2(Er)f_2(Eg_1)h_2\\
&+ r(E^{-1}s)^2(E^{-1}f_2)(E^{-1}g_2)h_1 - r(E^{-1}r)s(E^{-1}f_1)g_2h_1\\
&+ r(Er)(Es)(Ef_2)(Eg_1)h_1 - r(Er)(E^{-1}s)(Ef_1)(E^{-1}g_2)h_1\\
&+ r(E^{-1}r)(E^{-2}s)(E^{-2}f_2)(E^{-1}g_1)h_1 + r(E^{-1}r)(E^{-1}s)(E^{-1}f_1)(E^{-1}g_2)h_1\\
&- r(Er)sf_2(Eg_1)h_1 + r(Er)^2(Ef_1)(Eg_1)h_1 - r^2(E^{-1}r)f_1(E^{-1}g_1)h_1\\
&- r(E^{-1}r)(Er)(E^{-1}f_1)(Eg_1)h_1 + r(Er)(E^2r)(E^2f_1)(Eg_1)h_1\\
&- r^2(Er)f_1(Eg_1)h_1 + r(E^{-1}r)^2(E^{-1}f_1)(E^{-1}g_1)h_1 + r^2sf_1g_1h_2 - r^2sf_1g_2h_1\\
&+ r(E^{-1}r)(E^{-1}s)(E^{-1}f_1)(E^{-1}g_2)h_1 - r(Er)s(Ef_1)g_1h_2\\
&- (Er)rsf_1(Eg_1)h_2 + rs(Er)(Ef_1)g_2h_1 + (Er)^2s(Ef_1)(Eg_1)h_2\\
&- r^2(E^{-1}s)f_1(E^{-1}g_2)h_1 - r(Er)(E^{-1}r)(Ef_1)(E^{-1}g_1)h_1\\
&+ (E^{-1}r)(E^{-2}r)r(E^{-2}f_1)(E^{-1}g_1)h_1 + r(E^{-1}r)(E^{-1}s)(E^{-1}f_2)(E^{-1}g_1)h_1\\
&- r(E^{-1}r)(E^{-1}s)(E^{-1}f_2)(E^{-1}g_1)h_1](n)
\end{aligned}$$

$$\begin{aligned}
\sigma_2^2 : \sum_{n\in\mathbb{Z}}[&rf_2g_2h_1 - rf_2g_1h_2 + r(E^{-1}f_2)g_1h_2 - rf_2(E^{-1}g_2)h_1\\
&+ (Er)(Ef_2)(Eg_1)h_2 - r(E^{-1}f_2)g_2h_1 - (Er)f_2(Eg_1)h_2\\
&+ r(E^{-1}f_2)(E^{-1}g_2)h_1](n)
\end{aligned}$$

$$\begin{aligned}
\sigma_1\sigma_2 : \sum_{n\in\mathbb{Z}}[&rsf_2g_1h_2 - rsf_2g_2h_1 + r(Er)(Ef_1)g_1h_2 - r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_1\\
&+ (Er)sf_2(Eg_1)h_2 - r(E^{-1}s)(E^{-1}f_2)(E^{-1}g_2)h_1 - (Er)(Es)(Ef_2)(Eg_1)h_2\\
&+ rs(E^{-1}f_2)g_2h_1 - (Er)(E^2r)(E^2f_1)(Eg_1)h_2 + r(Er)(E^{-1}f_2)(Eg_1)h_1\\
&- r(E^{-1}r)(E^{-1}f_1)g_1h_2 + r(E^{-1}r)f_2(E^{-1}g_1)h_1 + r(Er)f_1(Eg_1)h_2\\
&- r(Er)(Ef_1)g_2h_1 - r(E^{-1}s)(E^{-1}f_2)g_1h_2\\
&+ r(E^{-1}s)f_2(E^{-1}g_2)h_1 - (Er)s(Ef_2)(Eg_1)h_2 + r(E^{-1}s)(E^{-1}f_2)g_2h_1\\
&+ (Er)sf_2(Eg_1)h_2 - r(E^{-1}s)(E^{-1}f_2)(E^{-1}g_2)h_1 + rsf_2g_1h_2\\
&- rsf_2g_2h_1 - rs(E^{-1}f_2)g_1h_2 + rsf_2(E^{-1}g_2)h_1\\
&+ r(E^{-1}r)(E^{-1}f_1)g_2h_1 - r(Er)(Ef_2)(Eg_1)h_1\\
&- r(Er)f_2(Eg_1)h_1 + r(Er)f_2(Eg_1)h_1 + r(Er)(Ef_1)(E^{-1}g_1)h_1\\
&- r(E^{-1}r)(E^{-2}f_2)(E^{-1}g_1)h_1 - r^2f_1g_1h_2
\end{aligned}$$

$$
\begin{aligned}
&+r^2 f_1 g_2 h_1 + r(Er)(Ef_1) g_1 h_2 - r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_1\\
&+r(Er) f_1 (Eg_1) h_2 - r(Er)(Ef_1) g_2 h_1 - (Er)^2 (Ef_1)(Eg_1) h_2\\
&+r^2 f_1 (E^{-1}g_2) h_1 - r(E^{-1}r)(E^{-1}f_2)(E^{-1}g_1)h_1\\
&+r(E^{-1}r)(E^{-1}f_2)(E^{-1}g_1)h_1](n)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1\sigma_3 : \sum_{n\in\mathbb{Z}} [&-(Er)(Es)(Ef_2)(Eg_2)h_2 + rsf_2(E^{-1}g_2)h_2 + (Er)sf_2(Eg_2)h_2\\
&-r(E^{-1}s)(E^{-1}f_2)(E^{-1}g_2)h_2 - (Er)(E^2 r)(E^2 f_1)(Eg_2)h_2\\
&+r(Er)(E^{-1}f_2)(Eg_1)h_2 + r(Er) f_1 (Eg_2)h_2 - r(Er)(Ef_2)g_2 h_1\\
&+r(Er)(Ef_1)(E^{-1}g_2)h_2 - r(E^{-1}r)(E^{-2}f_2)(E^{-1}g_2)h_1\\
&-r(E^{-1}r)(E^{-1}f_1)(E^{-1}g_2)h_2 + r(Er)(Ef_2)g_1 h_2\\
&+r^2(E^{-1}f_2)g_2 h_1 - (Er)^2(Ef_2)(Eg_1)h_2 + r^2 f_2(E^{-1}g_2)h_1\\
&-r^2(E^{-1}f_1)g_1 h_2](n)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2\sigma_3 : \sum_{n\in\mathbb{Z}} [&(Er)(Ef_2)(Eg_2)h_2 - rf_2(E^{-1}g_2)h_2\\
&-(Er) f_2(Eg_2)h_2 - r(E^{-1}f_2)(E^{-1}g_2)h_2](n).
\end{aligned}
$$

Through a careful check or a computer algebra system, we see that these five sums with a cycle of f, g and h are all equal to zero. It then follows that the Jacobi identity holds for the Hamiltonian operator (3.11).

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